

Note on Igusa's cusp form of weight 35

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Abstract

A congruence relation satisfied by Igusa's cusp form of weight 35 is presented. As a tool to confirm the congruence relation, a Sturm-type theorem for the case of odd-weight Siegel modular forms of degree 2 is included.

1 Introduction

In [5], Igusa gave a set of generators of the graded ring of degree 2 Siegel modular forms. In these generators, there are four even-weight forms $\varphi_4, \varphi_6, \chi_{10}, \chi_{12}$, and only one odd-weight form χ_{35} . Here φ_k is the normalized Eisenstein series of weight k , and χ_k is a cusp form of weight k .

The purpose of this paper is to introduce a strange congruence relation of the odd-weight cusp form X_{35} , which is a suitable normalization of χ_{35} (for the precise definition, see § 2.2).

Main result. Denote by $a(T; X_{35})$ the T -th Fourier coefficient of the cusp form X_{35} . If T satisfies $\det(T) \not\equiv 0 \pmod{23}$, then

$$a(T; X_{35}) \equiv 0 \pmod{23},$$

or equivalently,

$$\Theta(X_{35}) \equiv 0 \pmod{23},$$

where Θ is the theta operator on Siegel modular forms (for the precise definition, see § 2.5).

This result shows that *almost* all the Fourier coefficients $a(T; X_{35})$ are divisible by 23.

2 Preliminaries

2.1 Notation

First we confirm the notation. Let $\Gamma_n = Sp_n(\mathbb{Z})$ be the Siegel modular group of degree n and \mathbb{H}_n the Siegel upper-half space of degree n . We denote by $M_k(\Gamma_n)$ the \mathbb{C} -vector space of all Siegel modular forms of weight k for Γ_n , and $S_k(\Gamma_n)$ is the subspace of cusp forms.

Any $F(Z)$ in $M_k(\Gamma_n)$ has a Fourier expansion of the form

$$F(Z) = \sum_{T \in L_n} a(T; F) q^T, \quad q^T := e^{2\pi i \operatorname{tr}(TZ)}, \quad Z \in \mathbb{H}_n,$$

where T runs over all elements of L_n , and

$$\begin{aligned} \Lambda_n &:= \{T = (t_{ij}) \in \operatorname{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}, \\ L_n &:= \{T \in \Lambda_n \mid T \text{ is semi-positive definite}\}. \end{aligned}$$

In this paper, we deal mainly with the case of $n = 2$. For simplicity, we write

$$T = (m, n, r) \quad \text{for} \quad T = \begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix} \in \Lambda_2.$$

For a subring R of \mathbb{C} , let $M_k(\Gamma_n)_R \subset M_k(\Gamma_n)$ denote the space of all modular forms whose Fourier coefficients lie in R .

2.2 Igusa's generators

Let

$$M(\Gamma_2) = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)$$

be the graded ring of Siegel modular forms of degree 2. Igusa [5] gave a set of generators of the ring $M(\Gamma_2)$. The set consists of five generators

$$\varphi_4, \quad \varphi_6, \quad \chi_{10}, \quad \chi_{12}, \quad \chi_{35},$$

where φ_k is the normalized Eisenstein series on Γ_2 and χ_k is a cusp form of weight k . Moreover he showed that the even-weight generators $\varphi_4, \varphi_6, \chi_{10}, \chi_{12}$ are algebraically independent. Later, he extended the result to the integral case ([6]). Namely, he gave a minimal set of generators over \mathbb{Z} of the ring

$$M(\Gamma_2)_{\mathbb{Z}} = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}}.$$

The set of generators consists of fifteen modular forms including the following forms:

$$X_4 := \varphi_4, \quad X_6 := \varphi_6, \quad X_{10} := -2^{-2}\chi_{10}, \quad X_{12} := 2^2 \cdot 3\chi_{12}, \quad X_{35} := 2^2 i\chi_{35}.$$

Of course, these forms have rational integral Fourier coefficients under the following normalization:

$$\begin{aligned} a((0, 0, 0); X_4) &= a((0, 0, 0); X_6) = 1 \\ a((1, 1, 1); X_{10}) &= a((1, 1, 1); X_{12}) = 1 \\ a((2, 3, -1); X_{35}) &= 1. \end{aligned}$$

2.3 Order and the p -minimum matrix

We define a lexicographical order “ \succ ” for two different elements $T = (m, n, r)$ and $T' = (m', n', r')$ of Λ_2 by

$$\begin{aligned} T \succ T' &\iff (1) \operatorname{tr}(T) > \operatorname{tr}(T') \quad \text{or} \quad (2) \operatorname{tr}(T) = \operatorname{tr}(T'), \quad m > m' \quad \text{or} \\ &\quad (3) \operatorname{tr}(T) = \operatorname{tr}(T'), \quad m = m', \quad r > r'. \end{aligned}$$

Let p be a prime and $\mathbb{Z}_{(p)}$ the local ring consisting of p -integral rational numbers. For $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$, we define the p -minimum matrix $m_p(F)$ of F by

$$m_p(F) := \min\{ T \in L_2 \mid a(T; F) \not\equiv 0 \pmod{p} \},$$

where the “min” is defined in the sense of the above order. If $F \equiv 0 \pmod{p}$, then we define $m_p(F) = (\infty)$.

Remark 2.1. The p -minimum matrices of Igusa's generators are

$$m_p(X_4) = m_p(X_6) = (0, 0, 0), \quad m_p(X_{10}) = m_p(X_{12}) = (1, 1, -1), \quad m_p(X_{35}) = (2, 3, -1),$$

for any prime number p .

The following properties are essential.

- Lemma 2.2.** (1) $T_1 \succ T_2, S_1 \succ S_2$ implies $T_1 + S_1 \succ T_2 + S_2$.
(2) $T_1 \succ T_2$ implies $T_1 \pm S \succ T_2 \pm S$.
(3) $T + S = T' + S', T \succ T'$ implies $S \prec S'$.
(4) $m_p(F \cdot G) = m_p(F) + m_p(G)$.

Proof. (1), (2) Trivial.

(3) We use (2) without notice. By the assumption $T + S = T' + S'$, we have $T - T' = S' - S$. Then $0_2 \prec T - T' = S' - S$ because of $T \succ T'$. Hence $S \prec S'$.

(4) Let $m_p(F) = T_0$ and $m_p(G) = T'_0$. Then for all $T \prec T_0$ (resp. $T \prec T'_0$), $a(T; F) \equiv 0 \pmod{p}$ and $a(T_0; F) \not\equiv 0 \pmod{p}$ (resp. $a(T; G) \equiv 0 \pmod{p}$ and $a(T'_0; G) \not\equiv 0 \pmod{p}$). Now, recall that the T -th Fourier coefficient $a(T; F \cdot G)$ of $F \cdot G$ is given by

$$a(T; F \cdot G) = \sum_{\substack{S, S' \in L_2 \\ S + S' = T}} a(S; F)a(S'; G).$$

If $T \prec T_0 + T'_0$, then $T = S + S' \prec T_0 + T'_0$ and hence $S \prec T_0$ or $S' \prec T'_0$ because of (1). In this case, $a(S; F) \equiv 0 \pmod{p}$ or $a(S'; G) \equiv 0 \pmod{p}$. Therefore $a(S; F)a(S'; G) \equiv 0 \pmod{p}$ for each S, S' with $S + S' \prec T_0 + T'_0$. This implies $a(T; F \cdot G) \equiv 0 \pmod{p}$ for all $T \prec T_0 + T'_0$.

In order to complete the proof of (4), we need to prove that $a(T_0 + T'_0; F \cdot G) \not\equiv 0 \pmod{p}$. If $S + S' = T_0 + T'_0$, then we have by (3) that $S \prec T_0, S' \succ T'_0$ or $S \succ T_0, S' \prec T'_0$ or $S = T_0, S' = T'_0$. In the first two cases, since $a(S; F) \equiv 0 \pmod{p}$ or $a(S'; G) \equiv 0 \pmod{p}$, we get $a(S; F)a(S'; G) \equiv 0 \pmod{p}$. In the third case, $a(T_0; F)a(T'_0; G) \not\equiv 0 \pmod{p}$. Thus $a(T_0 + T'_0; F \cdot G) \not\equiv 0 \pmod{p}$, namely $m_p(F \cdot G) = T_0 + T'_0$. This completes the proof of (4). \square

2.4 Sturm-type theorem

A Sturm-type theorem for the Siegel modular forms of degree 2 was presented in [4]. We introduce the statement of this theorem for the case of level 1.

Theorem 2.3 (Choi-Choie-Kikuta [4]). Let p be a prime with $p \geq 5$ and k an even positive integer. For $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with Fourier expansion

$$F = \sum_{T \in L_2} a(T; F)q^T,$$

we assume that $a((m, n, r); F) \equiv 0 \pmod{p}$ for all m, n, r such that $0 \leq m, n \leq \frac{k}{10}$ and $4mn - r^2 \geq 0$. Then $F \equiv 0 \pmod{p}$.

We rewrite this theorem for later use:

Theorem 2.4. Let p be a prime with $p \geq 5$. Assume that $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ satisfies $m_p(F) \succ ([\frac{k}{10}], [\frac{k}{10}], r_0)$ for the maximum $r_0 \in \mathbb{Z}$ such that $([\frac{k}{10}], [\frac{k}{10}], r_0) \in L_2$. Then $m_p(F) = (\infty)$, i.e., $F \equiv 0 \pmod{p}$.

Proof. The assertion follows immediately from the inclusion

$$\left\{ T \in L_2 \mid T \preceq \left(\left[\frac{k}{10} \right], \left[\frac{k}{10} \right], r_0 \right) \right\} \supset \left\{ (m, n, r) \in L_2 \mid m, n \leq \frac{k}{10} \right\}. \quad (2.1)$$

\square

Remark 2.5. In general, the converse of inclusion (2.1) is not true. For example $([\frac{k}{10}] + 1, 0, 0) \prec ([\frac{k}{10}], [\frac{k}{10}], r_0)$ (for $k \geq 20$). We need a statement of this type to aid the proof of the next proposition.

In order to prove our main result, we need a Sturm-type theorem for *the odd-weight case*:

Proposition 2.6. Let p be a prime with $p \geq 5$ and k an odd positive integer. For $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$, we assume that $m_p(F) \succ ([\frac{k-35}{10}] + 2, [\frac{k-35}{10}] + 3, r_0 - 1)$, where $r_0 \in \mathbb{Z}$ is the maximum number such that $([\frac{k-35}{10}], [\frac{k-35}{10}], r_0) \in L_2$. Then $m_p(F) = (\infty)$, namely $F \equiv 0 \pmod{p}$.

Remark 2.7. When $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ is of odd weight, $X_{35} \cdot F \in M_{k+35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ is of even weight. Using Theorem 2.3 directly, we have the following statement: If $a((m, n, r); F) \equiv 0 \pmod{p}$ for all m, n, r such that $0 \leq m, n \leq \frac{k+35}{10}$ and $4mn - r^2 \geq 0$, then $F \equiv 0 \pmod{p}$.

For our purpose, however, the estimation of Proposition 2.6 is better than this estimation.

Proof of Proposition 2.6. First note that $M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} = X_{35}M_{k-35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ for odd k . Hence there exists $G \in M_{k-35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $F = X_{35} \cdot G$. Using (4) of Lemma 2.2, we get $m_p(F) = m_p(X_{35}) + m_p(G)$. Since $m_p(X_{35}) = (2, 3, -1)$, we have

$$m_p(G) = m_p(F) - (2, 3, -1) \succ \left(\left[\frac{k-35}{10} \right], \left[\frac{k-35}{10} \right], r_0 \right).$$

It should be noted that Lemma 2.2 (2) is used to get the last inequality. Since G is of even weight, we can apply Theorem 2.4 to G . This shows that $F = X_{35} \cdot G \equiv 0 \pmod{p}$. \square

2.5 Theta operator

In [3], Serre used the theta operator θ on elliptic modular forms to develop the theory of p -adic modular forms:

$$\theta = q \frac{d}{dq} : f = \sum a(t; f) q^t \mapsto \theta(f) := \sum t \cdot a(t; f) q^t.$$

Later the operator was generalized to the case of Siegel modular forms:

$$\Theta : F = \sum a(T; F) q^T \mapsto \Theta(F) := \sum \det(T) \cdot a(T; F) q^T$$

(e.g.cf. [3]). Moreover the following fact was proven:

Theorem 2.8 (Böcherer-Nagaoka [3]). Assume that a prime p satisfies $p \geq n + 3$. Then for any Siegel modular form F in $M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$, there exists a Siegel cusp form G in $S_{k+p+1}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ satisfying

$$\Theta(F) \equiv G \pmod{p}.$$

Example. Under the notation in § 2.2, we have

$$\Theta(X_6) \equiv 4X_{12} \pmod{5}.$$

3 Main result

On the basis of the previous preparation, we can now describe our main result.

Theorem 3.1. Let $a(T; X_{35})$ denote the Fourier coefficient of X_{35} . If $\det(T) \not\equiv 0 \pmod{23}$, then

$$a(T; X_{35}) \equiv 0 \pmod{23},$$

or equivalently,

$$\Theta(X_{35}) \equiv 0 \pmod{23}.$$

Proof. Our proof mainly depends on Proposition 2.6 and numerical calculation of the Fourier coefficients of X_{35} . If we use the theta operator, this assertion is equivalent to showing that

$$\Theta(X_{35}) \equiv 0 \pmod{23}.$$

From Theorem 2.8, there exists a Siegel cusp form $G \in S_{59}(\Gamma_2)_{\mathbb{Z}_{(23)}}$ such that

$$\Theta(X_{35}) \equiv G \pmod{23}.$$

Therefore the proof is reduced to showing that

$$G \equiv 0 \pmod{23}. \quad (3.1)$$

We now apply Proposition 2.6 to the form G . It then suffices to show that

$$a((m, n, r); G) \equiv 0 \pmod{23} \quad \text{for } T = (m, n, r) \text{ with } \text{tr}(T) = m + n \leq 10.$$

Since $a((m, n, r); G) = -a((n, m, r); G)$ for the odd-weight form G , this statement is equivalent to

$$a((m, n, r); \Theta(X_{35})) \equiv 0 \pmod{23} \quad \text{for } T = (m, n, r) \text{ with } \text{tr}(T) = m + n \leq 9.$$

We then write down the first part the Fourier expansion of X_{35} following the order introduced in § 2.3. For this, we set

$$q_{jk} := \exp(2\pi i z_{jk}) \quad \text{for } Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbb{H}_2.$$

The terms corresponding to $T = (m, n, r)$ with $\text{tr}(T) = m + n \leq 9$ are as follows:

$$\begin{aligned} X_{35} = & (q_{12}^{-1} - q_{12})q_{11}^2q_{22}^3 + (-q_{12}^{-1} + q_{12})q_{11}^3q_{22}^2 \\ & + (-q_{12}^{-3} - 69q_{12}^{-1} + 69q_{12} + q_{12}^3)q_{11}^2q_{22}^4 + (q_{12}^{-3} + 69q_{12}^{-1} - 69q_{12} - q_{12}^3)q_{11}^4q_{22}^2 \\ & + (69q_{12}^{-3} + 2277q_{12}^{-1} - 2277q_{12} - 69q_{12}^3)q_{11}^2q_{22}^5 \\ & + (q_{12}^{-5} - 32384q_{12}^{-2} - 129421q_{12}^{-1} + 129421q_{12} + 32384q_{12}^2 - q_{12}^5)q_{11}^3q_{22}^4 \\ & + (-q_{12}^{-5} + 32384q_{12}^{-2} + 129421q_{12}^{-1} - 129421q_{12} - 32384q_{12}^2 + q_{12}^5)q_{11}^4q_{22}^3 \\ & + (-69q_{12}^{-3} - 2277q_{12}^{-1} + 2277q_{12} + 69q_{12}^3)q_{11}^5q_{22}^2 \\ & + (q_{12}^{-5} - 2277q_{12}^{-3} - 47702q_{12}^{-1} + 47702q_{12} + 2277q_{12}^3 - q_{12}^5)q_{11}^2q_{22}^6 \\ & + (32384q_{12}^{-4} - 2184448q_{12}^{-2} - 3203072q_{12}^{-1} + 3203072q_{12} + 2184448q_{12}^2 - 32384q_{12}^4)q_{11}^3q_{22}^5 \\ & + (-32384q_{12}^{-4} + 2184448q_{12}^{-2} + 3203072q_{12}^{-1} - 3203072q_{12} - 2184448q_{12}^2 + 32384q_{12}^4)q_{11}^5q_{22}^3 \\ & + (-q_{12}^{-5} + 2277q_{12}^{-3} + 47702q_{12}^{-1} - 47702q_{12} - 2277q_{12}^3 + q_{12}^5)q_{11}^6q_{22}^2 \\ & + (-69q_{12}^{-5} + 47702q_{12}^{-3} + 709665q_{12}^{-1} - 709665q_{12} - 47702q_{12}^3 + 69q_{12}^5)q_{11}^2q_{22}^7 \\ & + (-q_{12}^{-7} + 129421q_{12}^{-5} + 2184448q_{12}^{-4} + 41321984q_{12}^{-2} + 105235626q_{12}^{-1} \\ & \quad - 105235626q_{12} - 41321984q_{12}^2 - 2184448q_{12}^4 - 129421q_{12}^5 + q_{12}^7)q_{11}^3q_{22}^6 \\ & + (-69q_{12}^{-7} - 32384q_{12}^{-6} + 107121810q_{12}^{-3} - 31380096q_{12}^{-2} + 759797709q_{12}^{-1} \\ & \quad - 759797709q_{12} + 31380096q_{12}^2 - 107121810q_{12}^3 + 32384q_{12}^6 + 69q_{12}^7)q_{11}^4q_{22}^5 \\ & + (69q_{12}^{-7} + 32384q_{12}^{-6} - 107121810q_{12}^{-3} + 31380096q_{12}^{-2} - 759797709q_{12}^{-1} \\ & \quad + 759797709q_{12} - 31380096q_{12}^2 + 107121810q_{12}^3 - 32384q_{12}^6 - 69q_{12}^7)q_{11}^5q_{22}^4 \\ & + (q_{12}^{-7} - 129421q_{12}^{-5} - 2184448q_{12}^{-4} - 41321984q_{12}^{-2} - 105235626q_{12}^{-1} \\ & \quad + 105235626q_{12} + 41321984q_{12}^2 + 2184448q_{12}^4 + 129421q_{12}^5 - q_{12}^7)q_{11}^6q_{22}^3 \\ & + (69q_{12}^{-5} - 47702q_{12}^{-3} - 709665q_{12}^{-1} + 709665q_{12} + 47702q_{12}^3 - 69q_{12}^5)q_{11}^7q_{22}^2 + \cdots \end{aligned}$$

The Fourier coefficients different from ± 1 are as follows:

$$a((4, 1, 2); X_{35}) = -69 = -3 \cdot \underline{23}, \quad a((5, 1, 2); X_{35}) = 2277 = 3^2 \cdot 11 \cdot \underline{23},$$

$$\begin{aligned}
a((4, 1, 3); X_{35}) &= -1294121 = -17 \cdot \underline{23} \cdot 331, & a((4, 2, 3); X_{35}) &= -32384 = -2^7 \cdot 11 \cdot \underline{23}, \\
a((6, 1, 2); X_{35}) &= -47702 = -2 \cdot 17 \cdot \underline{23} \cdot 61, & a((5, 1, 3); X_{35}) &= -3203072 = -2^{13} \cdot 17 \cdot \underline{23}, \\
a((5, 2, 3); X_{35}) &= -2184448 = -2^8 \cdot 7 \cdot \underline{23} \cdot 53, & a((7, 1, 2); X_{35}) &= 709665 = 3 \cdot 5 \cdot 11^2 \cdot 17 \cdot \underline{23}, \\
a((6, 1, 3); X_{35}) &= 105235626 = 2 \cdot 3 \cdot \underline{23} \cdot 762577, & a((6, 2, 3); X_{35}) &= 41321984 = 2^9 \cdot 11^2 \cdot \underline{23} \cdot 29, \\
a((5, 1, 4); X_{35}) &= 759797709 = 3 \cdot 11 \cdot \underline{23} \cdot 29 \cdot 34519, & a((5, 2, 4); X_{35}) &= -31380096 = -2^7 \cdot 3 \cdot 11 \cdot 17 \cdot 19 \cdot \underline{23}, \\
a((5, 3, 4); X_{35}) &= 107121810 = 2 \cdot 3 \cdot 5 \cdot 19 \cdot \underline{23} \cdot 8171.
\end{aligned}$$

All of these Fourier coefficients are divisible by 23. On the other hand, if $a(T; X_{35}) = \pm 1$ for T in this range, then $\det(T) = 23/4 \equiv 0 \pmod{23}$. This fact implies that

$$a((m, n, r); \Theta(X_{35})) \equiv 0 \pmod{23}$$

for $T = (m, n, r)$ with $\text{tr}(T) = m + n \leq 9$. Therefore, we obtain

$$a((m, n, r); G) \equiv 0 \pmod{23}$$

for $T = (m, n, r)$ with $\text{tr}(T) = m + n \leq 9$. Consequently we have (3.1). This completes the proof of our theorem. \square

Remark 3.2. (1) The converse statement of the theorem is not true in general. In fact

$$a((1, 6, 1); X_{35}) = 0 \quad \text{and} \quad \det((1, 6, 1)) = 23/4 \equiv 0 \pmod{23}.$$

Numerical examples of $a((m, n, r); X_{35})$ are found in [1], page 277.

(2) There are other “modulo 23” congruences for the Siegel modular forms in [2], Satz 5,(a). In that case, the congruence is concerned with the Eisenstein lifting of the Ramanujan delta function.

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